MMAT 5010 Linear Analysis (2024-25): Homework 9
Deadline: 19 Apr 2025

Important Notice:

♣ The answer paper must be submitted before the deadline.
♠ The answer paper MUST BE sent to the CU Blackboard. Please refer to the course web for details.

1. Let $(X, \langle\cdot,\cdot\rangle)$ be an inner product space. Show that the inner product $\langle\cdot,\cdot\rangle : X \times X \longrightarrow \mathbb{C}$ is continuous, that is, whenever the sequences $x_n \to x$ and $y_n \to y$ in X, we have $\langle x_n, y_n\rangle \to \langle x, y\rangle$.
   From this show that if A is a subset of X, then $A^\perp := \{x \in X : x \perp y, \text{for all } y \in A\}$ is a closed subset of X.

2. Let $(X, \langle\cdot,\cdot\rangle_X)$ and $(Y, \langle\cdot,\cdot\rangle_Y)$ be Hilbert spaces. For $(x_1, y_1), (x_2, y_2) \in X \times Y$, put

$$\langle (x_1, y_1), (x_2, y_2)\rangle_{X\times Y} := \langle x_1, x_2\rangle_X + \langle y_1, y_2\rangle_Y.$$

   Show that $\langle\cdot,\cdot\rangle_{X\times Y}$ is an inner product on the direct sum $X \times Y$ and it is a Hilbert space under this inner product.

3. Let X be a Hilbert space and let $T, S \in B(X)$. Show that
   (a) $(TS)^* = S^*T^*$.
   (b) if T is invertible, that in $T^{-1} \in L(X)$ exists, then $(T^{-1})^* = (T^*)^{-1}$.

$* * *$ **End** $* * *$